Bessel moments, random walks and Calabi–Yau equations

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Abstract

I prove a theorem that relates moments of Bessel functions to integrals recently considered in the context of random walks. Strong support is found, at 50 digit precision, for a conjecture that had been based on scant data. The recursions used in this work lead to Calabi–Yau differential equations, with maximal unipotent monodromy, for the Green functions of generalizations of the diamond lattice to $D > 3$ spatial dimensions. Studying these for $D < 10$, I am led to conjectures on the Yukawa couplings associated with their mirror maps. A suggestion that the face centred cubic lattice in $D = 5$ dimensions might lead to a Calabi–Yau equation is not borne out by detailed calculation.

1 Introduction

In a recent study [10] of the n-dimensional random walk integral

$$W_n(s) = \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 \mathrm{d}x_n \left| \sum_{k=1}^{n} \exp(2\pi \mathrm{i} x_k) \right|^s \tag{1}$$

it was conjectured that

$$W_{2n}(s) \overset{?}{=} \sum_{j \geq 0} \binom{s/2}{j}^2 W_{2n-1}(s-2j) \tag{2}$$

for positive integers n and s. For even s this is easy to prove; for odd s the evidence given in [10] was scanty, with only three digits of numerical precision achieved for a few cases of (2) with $n = 3$.

In Section 2, I show that (1), with odd s, may be evaluated as a moment of Bessel functions. In Section 3, I achieve 50 digits of precision for testing all 100 cases of (2) with $n = 2, 3, 4, 5$ and odd $s < 50$. The conjecture emerges intact.

The recursion in s for $W_n(s)$ leads to the differential equation for the Green function of a diamond lattice in $D = n - 1$ dimensions. For $D = 4$ this has a Calabi–Yau [3, 4]

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form and for $D = 5$ it can be pulled back [3] to such a form. In Section 4, I investigate the mirror maps and Yukawa couplings [5] of the cases with $D < 10$ and am led to two conjectures. In $D = 4$ dimensions, it was recently shown [13] that the face centred cubic (FCC) lattice also leads to a Calabi–Yau equation. I investigate a suggestion that the FCC lattice in $D = 5$ dimensions might permit pull back to a fourth order Calabi-Yau equation. This does not appear to be the case. Instead, I provide a differential equation of order 6 and degree 12 that determines this Green function. This equation lacks maximal unipotent monodromy. Section 5 provides some comments.

2 Evaluation of $W_n(2k-1)$

Theorem 1: For integers $n > 2$ and $k \geq 0$,

$$W_n(2k-1) = \frac{(2k)!}{2^k k!} \int_0^\infty \mathrm{d}x \left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k J_0^n(x). \tag{3}$$

Proof: From the work of Kluyver [15], more than a century ago, it follows that (1) is given, for any limit of integration $b \geq n$, by the moment [10]

$$W_n(s) = \int_0^b \mathrm{d}t\, t^s p_n(t) \tag{4}$$

of the probability distribution

$$p_n(t) = \int_0^\infty \mathrm{d}x\, xt J_0(xt) J_0^n(x). \tag{5}$$

For an account of this, 50 years later, see [12]. Next, I prove by induction that

$$p_n(t) = \frac{1}{t^{2k}} \int_0^\infty \mathrm{d}x\, (xt)^{k+1} J_k(xt) \left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k J_0^n(x) \tag{6}$$

for any integer $k \geq 0$, using the Bessel function identity

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z^k J_k(z)\right) = z^k J_{k-1}(z) \tag{7}$$

and integration by parts. I substitute (6) in (4), take the limit $b \to \infty$, set $z = xt$ and use the standard Bessel integral [1]

$$\int_0^\infty \mathrm{d}z\, z^{s+1-k} J_k(z) = 2^{s+1-k} \frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(k-\frac{s}{2}\right)} \tag{8}$$

to obtain

$$W_n(s) = 2^{s+1-k} \frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(k-\frac{s}{2}\right)} \int_0^\infty \frac{\mathrm{d}x}{x^{s+1-2k}} \left(-\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k J_0^n(x) \tag{9}$$

which is valid for real s with $2k > s > \max(-2, -\frac{n}{2})$, where the Gamma functions have positive arguments and the integrals converge. Finally, I set $s = 2k-1$ and hence prove (3) for integers $n > 2$ and $k \geq 0$, by reducing Gamma functions to factorials.

Remark 1: Apart from an error in overall sign, a special case of (9), at $k = 0$, was given in [10]. Unfortunately, this case is of no use for computing $W_n(s)$ at positive s. Instead, the authors of [10] resorted to cumbersome integrals involving both Struve functions and Bessel functions, which provided only low precision for checking conjecture (2). Thanks to the theorem, I am now able to investigate this conjecture much more vigourously.

3 Investigation of the conjecture

For $n = 1$, conjecture (2) is clearly true, since $W_1(s) = 1$ and

$$W_2(s) = \binom{s}{s/2} = \sum_{j\ge 0} \binom{s/2}{j}^2 . \tag{10}$$

It is also straightforward to prove (2) for even positive integers $s = 2k$, since $W_n(2k)$ is an integer that enumerates self returning walks of length $2k$ on the generalization of a three–dimensional diamond lattice to $D = n - 1$ spatial dimensions. To show this, I increase k to $k + 1$ in (9), set $s = 2k$ and evaluate the resulting integral of a differential in terms of a Taylor coefficient

$$W_n(2k) = k! \left(-\frac{2}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^k J_0^n(x)\Bigg|_{x=0} \tag{11}$$

at the lower limit of integration. Then the nth power of the expansion

$$J_0(x) = \sum_{j\ge 0} \frac{(-x^2/4)^j}{(j!)^2} \tag{12}$$

gives a sum of squares of multinomial coefficients in

$$W_n(2k) = \sum_{j_1+\ldots+j_n=k} \left(\frac{k!}{j_1!\ldots j_n!}\right)^2 = \sum_{j=0}^{k} \binom{k}{j}^2 W_{n-1}(2k-2j) \tag{13}$$

with a recursion that clearly guarantees (2), for even $s = 2k$. For the roles of the integers $W_n(2k)$, in physics and mathematics, see [14] and [17], respectively.

By contrast, conjecture (2) received scant numerical evidence for odd s in [10]. For example, at $n = 3$ it was checked to only three digits of numerical precision, for the three cases with $s = 1, 3, 5$. I now subject this conjecture to much more rigorous scrutiny, at a precision of more than 50 decimal digits, for all of the 100 cases with $1 < n < 6$ and odd $s < 50$.

3.1 Recursions for $W_n(s)$ with $n < 7$

The differential equation for $J_0^n(x)$ yields a recursion for $W_n(s)$ of the form

$$s^{n-1} W_n(s) + \sum_{j=1}^{\lceil n/2 \rceil} (-1)^j P_{n,j}(s-j) W_n(s-2j) = 0 \tag{14}$$

where $P_{n,j}(x) = (-1)^{n-1} P_{n,j}(-x)$, with $2j < n + 2$, is a polynomial of degree $n - 1$. In particular, $P_{1,1}(x) = 1$ and $P_{2,1}(x) = 4x$. Then the polynomials

$$P_{3,1}(x) = 10x^2 + 2,\ P_{3,2}(x) = 9x^2,\ P_{4,1}(x) = 20x^3 + 12x,\ P_{4,2}(x) = 64x^3, \tag{15}$$

$$P_{5,1}(x) = 35x^4 + 42x^2 + 3,\ P_{5,2}(x) = 259x^4 + 104x^2,\ P_{5,3}(x) = (15(x^2-1))^2, \tag{16}$$

$$P_{6,1}(x) = 8x(7x^4 + 14x^2 + 3),\ P_{6,2}(x) = 16x^3(49x^2 + 59),\ P_{6,3}(x) = x(48(x^2-1))^2 \tag{17}$$

compactly encode the recursions given in [8, 14] for $n < 7$.

3.2 Status of the conjecture for $W_4(2k-1)$

It is an easy matter to compute $W_3(2k-1)$ to very high precision, using the process of the arithmetic–geometric mean (AGM) [9], since

$$W_3(-1) = \int_0^\infty \mathrm{d}x\, J_0^3(x) = \frac{4}{\pi^3}\int_0^\infty \mathrm{d}x\, K_0^3(x) = \frac{\sqrt{3}}{2}\left(\frac{1}{\mathrm{agm}\,(1,\cos(\pi/12))}\right)^2 \tag{18}$$

evaluates to a product of elliptic integrals at the third singular value, yielding 10,000 decimal digits in a mere tenth of a second. Note the amusing relation in (18) to a moment in [6] of a Bessel function K_0 of the second kind. Much more generally, Wilfrid Norman Bailey [7] evaluated, more than 70 years ago, integrals of products of three distinct Bessel functions. By such methods, one obtains an evaluation of

$$W_3(1) = 3\int_0^\infty \frac{\mathrm{d}x}{x}\, J_0^2(x)J_1(x) = W_3(-1) + \frac{6}{\pi^2 W_3(-1)}. \tag{19}$$

To test the conjecture at $n = 2$, one needs evaluations of $W_4(-1)$ and $W_4(1)$, from which the values of $W_4(2k-1)$ are then easily obtained, using the recursion relation. In less than three minutes, I obtained more than 50 digits of

$$\begin{aligned} W_4(-1) &= \int_0^\infty \mathrm{d}x\, J_0^4(x) \\ &= 0.90272857832383482419039339877276298503046350360141\ldots && (20) \\ W_4(1) &= 4\int_0^\infty \frac{\mathrm{d}x}{x}\, J_0^3(x)J_1(x) \\ &= 1.79909247984285103353260284584610891006628200329162\ldots && (21) \end{aligned}$$

by Taylor expansions of the integrands at small x, asymptotic expansions at large x and numerical quadrature in between, enabled by Pari–GP, running on a single core of an AMD64 machine. By contrast, only 6 good digits of $W_4(1)$ were recorded in Table 1 of [10], notwithstanding the availability of 256 cores at the Lawrence Berkeley National Laboratory.

The evidence in favour of the conjecture at $n = 2$ is now compelling. Using the Bessel moments (20,21) and the arithmetic–geometric mean in (18,19), I have confirmed (2) for $W_4(s)$ at 50 digit precision for every odd positive integer $s < 50$. I remark that if (2) is true, then 1000 digits of (20,21) may be obtained in 10 seconds.

3.3 Status of the conjecture for $W_6(2k-1)$

In a few minutes, I obtained more than 50 digits of

$$\begin{aligned} W_5(-1) &= 0.75360399902684225215051501541337137652515856330051\ldots && (22) \\ W_5(1) &= 2.00816184541542457073453536461419848894193033431120\ldots && (23) \\ W_5(3) &= 14.28958551898214988405421477294723606525430278244 64\ldots && (24) \end{aligned}$$

from (3) in Theorem 1. Then the recursion (14), with the polynomials in (16,17), gives a conjectured evaluation of $W_6(2k-1)$ as a linear combination of these three constants, with coefficients that are infinite sums with rational summands. I computed truncations of these sums, at $j = N$, for 51 values of $N > 500$, and fitted them with polynomials in $1/N$, of degree 50, obtaining at least 60 good digits for the limit $N \to \infty$. Then, for $k = 0, 1, 2$, I compared $W_6(2k-1)$, as predicted by the conjecture, with the direct evaluations

$$\begin{aligned} W_6(-1) &= 0.70642698445719203724945229878374471115937661897050\ldots && (25)\\ W_6(1) &= 2.19385900355176468534103120831101192964629504601816\ldots && (26)\\ W_6(3) &= 18.91326570469288593013883452738473634704862400047 40\ldots && (27) \end{aligned}$$

from Theorem 1. Agreement was found to more than 50 digits. More generally, I have confirmed (2) for $W_6(s)$ at 50 digit precision for every odd positive integer $s < 50$. Given the 6 moments in (22) to (27), this takes less than 10 seconds. I remark that Table 1 of [10] is unreliable. For example, $W_6(9) = 82718.8883930292923425396\ldots$ was recorded in [10] as 82718.498638208, with 9 bad digits after the decimal point.

3.4 Status of the conjecture for $W_8(2k-1)$

I computed more than 50 digits of the Bessel moments

$$\begin{aligned} W_7(-1) &= 0.65057178525749227031273893879241547354405455285481\ldots && (28)\\ W_7(1) &= 2.36637222333622475633264985600901982055877557620917\ldots && (29)\\ W_7(3) &= 23.94766413826122253276213446592181282867614256246 68\ldots && (30)\\ W_7(5) &= 374.74814636968291449293822869385776924316187637 6648\ldots && (31) \end{aligned}$$

which may be extended by recursion (14), using the polynomials

$$P_{7,1}(x) = 4(21x^6 + 63x^4 + 27x^2 + 1),\ P_{7,2}(x) = 6x^2(329x^4 + 792x^2 + 136), \qquad (32)$$

$$P_{7,3}(x) = 4(x^2-1)^2(3229x^2 + 1949),\ P_{7,4}(x) = (105x(x^2-4))^2. \qquad (33)$$

Then conjecture (2) at $n = 4$ indeed reproduces the Bessel moments

$$\begin{aligned} W_8(-1) &= 0.61263875088454441774988431976574663450628062326321\ldots && (34)\\ W_8(1) &= 2.52665658411243915692179057089373375296116648741071\ldots && (35)\\ W_8(3) &= 29.36289971039481775372114453794662042923573535013 14\ldots && (36)\\ W_8(5) &= 532.80362725572132493702295038353914657289534099 8276\ldots && (37) \end{aligned}$$

at 50 digit precision. Extending these with the polynomials

$$P_{8,1}(x) = 8x(15x^6 + 63x^4 + 45x^2 + 5),\ P_{8,2}(x) = 48x^3(91x^4 + 365x^2 + 188), \qquad (38)$$

$$P_{8,3}(x) = 256x(x^2-1)^2(205x^2 + 371),\ P_{8,4}(x) = x(384x(x^2-4))^2, \qquad (39)$$

I confirmed (2) for $W_8(s)$ at 50 digit precision for every odd positive integer $s < 50$.

3.5 Status of the conjecture for $W_{10}(2k-1)$

I computed more than 50 digits of the Bessel moments

$$
\begin{aligned}
W_9(-1) &= 0.57829144272849314519620791214901799393309322251236\ldots && (40)\\
W_9(1) &= 2.6775677111512081409060968891896944915211567229038 9\ldots && (41)\\
W_9(3) &= 35.133454956320923746486260319515940388972197261599 1\ldots && (42)\\
W_9(5) &= 725.26412831123558595608471657682467447458540432717 8\ldots && (43)\\
W_9(7) &= 19767.463565956469930637048882537663501227582752200 3\ldots && (44)
\end{aligned}
$$

which may be extended by recursion (14), using the polynomials

$$
\begin{aligned}
P_{9,1}(x) &= 165x^8+924x^6+990x^4+220x^2+5, && (45)\\
P_{9,2}(x) &= 6x^2(1463x^6+8800x^4+9064x^2+896), && (46)\\
P_{9,3}(x) &= 2(x^2-1)^2(86405x^4+312598x^2+85149), && (47)\\
P_{9,4}(x) &= 9x^2(x^2-4)^2(117469x^2+94664), && (48)\\
P_{9,5}(x) &= (945(x^2-1)(x^2-9))^2. && (49)
\end{aligned}
$$

Then conjecture (2) at $n=5$ indeed reproduces the Bessel moments

$$
\begin{aligned}
W_{10}(-1) &= 0.55007672739830097792009126393089394380627775605833\ldots && (50)\\
W_{10}(1) &= 2.8203538275437104778963913433993420690451000089552 4\ldots && (51)\\
W_{10}(3) &= 41.238855896526766201771714164595817566153396072125 2\ldots && (52)\\
W_{10}(5) &= 954.31784394770515177271120804849767780594087950175 8\ldots && (53)\\
W_{10}(7) &= 29337.281155508431439033700263927222987349670399034 0\ldots && (54)
\end{aligned}
$$

at 50 digit precision. Extending these with the polynomials

$$
\begin{aligned}
P_{10,1}(x) &= 4x(55x^8+396x^6+594x^4+220x^2+15), && (55)\\
P_{10,2}(x) &= 16x^3(1023x^6+8613x^4+14784x^2+4384), && (56)\\
P_{10,3}(x) &= 64x(x^2-1)^2(7645x^4+46079x^2+37644), && (57)\\
P_{10,4}(x) &= 1024x^3(x^2-4)^2(5269x^2+12731), && (58)\\
P_{10,5}(x) &= x(3840(x^2-1)(x^2-9))^2, && (59)
\end{aligned}
$$

I confirmed (2) for $W_{10}(s)$ at 50 digit precision for every odd positive integer $s<50$.

4 Calabi–Yau differential equations

With $n>1$ and $\theta = z\,\mathrm{d}/\mathrm{d}z$, the recursion (14) provides the differential equation

$$
\left[(2\theta)^{n-1}+\sum_{j=1}^{\lceil n/2\rceil}(-z)^jP_{n,j}(2\theta+j)\right]G_n(z)=0 \qquad (60)
$$

for the Green function

$$
G_n(z)=\sum_{k\ge 0}W_n(2k)z^k=1+nz+n(2n-1)z^2+n(6n^2-9n+4)z^3+O(z^4) \qquad (61)
$$

which is the unique solution to (60) with $G_n(0) = 1$. For $n = 2$, the solution is simply $G_2(z) = 1/\sqrt{1-4z}$. For $n = 3$, the regular solution is an elliptic integral that is very rapidly computable by the process of the arithmetic–geometric mean, with [6]

$$G_3(y^2) = \frac{1}{\text{agm}\left(\sqrt{(1-3y)(1+y)^3}, \sqrt{(1+3y)(1-y)^3}\right)} \tag{62}$$

giving the Green function for a hexagonal lattice, in $D = 2$ spatial dimensions. I remark that Pari–GP neatly returns the series expansion

$$G_3(y^2) = 1 + 3y^2 + 15y^4 + 93y^6 + 639y^8 + 4653y^{10} + 35169y^{12} + 272835y^{14} + O(y^{16}) \tag{63}$$

when asked to evaluate the right hand side of (62) for arbitrary y. The coefficient of y^{2k} is the number of ways a bee may take a walk of $2k$ steps in a plane of a honeycomb and return to where she started.

Let the height h of a self returning walk be the largest number of steps from the origin, during the walk. On a hexagonal lattice, there are $W_3(2) = 3$ two–step walks, each with $h = 1$, and $W_3(4) = 3^2 + 6 = 15$ four–step walks, with 6 of these having $h = 2$. It is already quite demanding to count the $W_3(6) = 3^3 + 48 + 18 = 93$ self returning walks with 6 steps. Of these, 3^3 clearly have $h = 1$. The 48 walks with $h = 2$ comprise 36 in which the bee revisits the origin after two or four steps and 12 in which she returns only after 6 steps. The latter 12 walks comprise 6 in which she revisits a site with $h = 2$ and 6 in which she visits two such sites. Finally, the 18 walks with $h = 3$ comprise 12 with a reversal of direction, after three steps, and 6 in which the bee traverses a hexagon of the lattice. I find it remarkable that one may effect such delicate enumerations by taking the reciprocal of an AGM in (62) and simply wonderful that the complementary AGM figures in decays into three particles in quantum field theory [6, 11].

4.1 Diamond lattice Green function

It was the capital discovery of Geoffrey Joyce [16] that the Green function G_4 of the diamond lattice, in $D = 3$ dimensions, may be obtained from the square of G_3 by a quadratic transformation of variables. This may be written rather simply as [6]

$$G_4(z) = (1-y)(1-9y)G_3^2(y), \quad \text{for} \quad z = -\frac{y}{(1-y)(1-9y)}. \tag{64}$$

Once discovered, this beautiful result is easily proved by computer algebra. One may use the polynomials in (15) to prove that the stated rational transformation from z to y ensures that $G_4(z)$ and $(1-y)(1-9y)G_3^2(y)$ satisfy the same third order differential equation in y. Then the expansion (61) shows that $G_4(z) - (1-y)(1-9y)G_3^2(y) = O(y^4)$ and hence that this combination vanishes for all y.

4.2 Mirror maps and Yukawa couplings

For a differential equation of the form (60), with order $n - 1 > 3$, we may define a mirror map and Yukawa coupling [5] by studying the first three elements, $y_0(z)$, $y_1(z)$ and $y_2(z)$, of the Frobenius basis of solutions. Here $y_0(z)$ is the regular solution, with

$y_0(0) = 1$. Then $y_1(z)$ and $y_2(z)$ are defined by requiring that $y_1(z) - y_0(z)\log(z)$ and $y_2(z) - y_1(z)\log(z) + \frac{1}{2}y_0(z)\log^2(z)$ are regular at $z = 0$. These solutions exist and are unique, since (60) has maximal unipotent monodromy [5] (MUM). The mirror map $q \to z(q)$ is the inverse of $z \to q(z) = \exp(y_1(z)/y_0(z))$. Then one defines a Yukawa coupling

$$K(q) = \left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 \frac{y_2(z(q))}{y_0(z(q))} \tag{65}$$

which is extremely robust under transformations of the original differential equation that may result from changes of variable and rescalings of solutions, such as occur in (64). Finally one extracts a sequence of numbers n_k from the Lambert series

$$K(q) = 1 + \sum_{k>0} \frac{n_k q^k}{1-q^k}. \tag{66}$$

In [5], the authors studied Calabi–Yau equations with the properties that

(a) each is a fourth order differential equation with MUM,

(b) the exterior square has order 5,

(c) the expansion of $y_0(z)$ has integer coefficients,

(d) the expansion of $q(z) = \exp(y_1(z)/y_0(z))$ has integer coefficients,

(e) there is a small integer N_0 such that $N_k = N_0 n_k/k^3$ is an integer for $k > 0$.

Here, the exterior square is the differential equation satisfied by the Wronskian of any two solutions of the fourth order equation. Generically, this has order 6. So the restriction to an exterior square with the lesser order 5 implies a condition on the coefficients of the original equation. In [3], it is conjectured that conditions (a), (c) and (d) are sufficient to ensure properties (b) and (e). Appendix C of the tabulation [4] provides a "superseeker" index of values of N_0, $|N_1|$ and $|N_3|$ for known fourth order Calabi–Yau equations.

4.3 Lattice Green functions in four dimensions

The fourth order differential equation for the Green function $G_5(z)$ of a diamond lattice in $D = 4$ spatial dimensions is recorded as entry #34 in [4], with superseeker integers $N_0 = 3$, $|N_1| = 3$, $|N_3| = 28$.

Generalizations of cubic lattices to 4 dimensions yield Green functions whose fourth order differential equations satisfy the desiderata of [5]. For the body centred cubic (BCC) and simple cubic (SC) lattices, one obtains entries #3 and #16 in [4]. Recently, Tony Guttmann [13] obtained a differential equation for the face centred cubic (FCC) lattice in 4 dimensions, by empirical methods, taking several hours to compute 40 expansion coefficients. It is recorded as entry #366 in the latest update of [4], with $N_0 = 1$, $|N_1| = 3$, $|N_3| = 64$.

In fact, it takes only a few seconds to recover and simplify the differential equation for the FCC lattice Green function with $D = 4$, using its series expansion

$$F_4(z) = \sum_{k\geq 0} k!\, z^k \sum_{j_0+\ldots+j_5=k} \frac{S_{0,1,2}\, S_{0,3,4}\, S_{1,3,5} S_{2,4,5}}{j_0!j_1!j_2!j_3!j_4!j_5!} \tag{67}$$

with $S_{a,b,c}$ taking the value $\binom{2s}{s}$ if $s = (j_a + j_b + j_c)/2$ is an integer and 0 otherwise. Evaluating 5–fold sums for the expansion coefficients, I obtained a differential equation for $\widetilde{F}_4(z) = F_4(z/(1-18z))/(1-18z)$ of the Calabi–Yau form

$$\left[(2\theta)^4 + \sum_{j=1}^{6}(-z)^j P_j(2\theta + j)\right] \widetilde{F}_4(z) = 0 \tag{68}$$

with degree 6 and even polynomials

$$P_1(x) = 105x^4 + 166x^2 + 17, \ P_2(x) = 2(2095x^4 + 2912x^2 + 432), \tag{69}$$

$$P_3(x) = 72(1155x^4 - 892x^2 + 577), \ P_4(x) = 864(1011x^4 - 5059x^2 + 4900), \tag{70}$$

$$P_5(x) = 75600(x^2 - 9)(61x^2 - 145), \ P_6(x) = 9525600(x^2 - 4)(x^2 - 16) \tag{71}$$

yielding singularities for $1/z = 0, 6, 10, 14, 15, 18, 42$. The mirror map for the differential equation (68) gives a Yukawa coupling $K(q)$ whose instanton numbers, n_k/k^3, are

$$3, -4, 64, -253, 4292, -25608, 442008, -3202512, 56565002, -457852636 \tag{72}$$

for $k = 1 \ldots 10$. These are the same as may be obtained from the less compact equation of degree 7 given in [13]. I have verified that (68) reproduces the series expansion (67) up to z^{100} and that n_k/k^3 is an integer up to $k = 100$.

4.4 Lattice Green functions in 5 dimensions

It was shown in [2] how to pull back the fifth order equation, with degree three in z, for the Green function $G_6(z)$ of a diamond lattice in $D = 5$ spatial dimensions, to a fourth order equation, with degree 6, yielding the integers $N_0 = 6$, $|N_1| = 12$, $|N_3| = 140$ of entry #130 in [4].

It is notable that only one other example of pullback from order 5 and degree greater than two is know. That derives from the SC lattice in 5 dimensions, with a Green function $\sum_{k\geq 0}\binom{2k}{k}W_5(2k)z^k$, whose fifth order differential equation, with degree three, also has a pullback to order four and degree 6, yielding the integers $N_0 = 3$, $|N_1| = 24$, $|N_3| = 1552$ of entry #188 in [4].

It was suggested to me by Tony Guttmann that the FCC lattice Green function in 5 dimensions might yield a fifth order equation with a pullback to a Calabi–Yau equation of order four. Finding such an equation appeared to be a daunting task, since the explicit expansion is

$$F_5(z) = \sum_{k\geq 0} k!\, z^k \sum_{j_0+\ldots+j_9=k} \frac{T_{0,1,2,3}\, T_{0,4,5,6}\, T_{1,4,7,8}\, T_{2,5,7,9}\, T_{3,6,8,9}}{j_0!j_1!j_2!j_3!j_4!j_5!j_6!j_7!j_8!j_9!} \tag{73}$$

with $T_{a,b,c,d}$ taking the value $\binom{2t}{t}$ if $t = (j_a + j_b + j_c + j_d)/2$ is an integer and 0 otherwise. Thus each expansion coefficient is given by a 9 fold sum. After several CPU days, I was able to obtain enough data to conclude that the differential equation is, unfortunately, of order 6, with degree 13, and lacks MUM. The degree may be reduced to 12 by working with $\widetilde{F}_5(z) = F(z/(1-8z))/(1-8z)$, whose differential equation has the form

$$\left[3^4\theta^5(\theta - 1) + \sum_{j=1}^{12} z^j Q_j(\theta)\right] \widetilde{F}_5(z) = 0 \tag{74}$$

where $\theta = z\,\mathrm{d}/\mathrm{d}z$ and the 12 polynomials Q_j are given here, in Pari–GP format:

```
Q1(x)=-2*3^3*x*(478*x^5-515*x^4+366*x^3+234*x^2+81*x+12);
Q2(x)=2^2*3^3*(21670*x^6-4614*x^5+22013*x^4+7456*x^3+555*x^2+28*x+128);
Q3(x)=-2^4*3*(2077018*x^6-204823*x^5+1814868*x^4+31347*x^3\
-785268*x^2-559440*x-144864);
Q4(x)=2^5*(69192712*x^6-50419976*x^5-216097437*x^4-447047910*x^3\
-457176285*x^2-252964860*x-60146208);
Q5(x)=-2^7*(122458544*x^6-955038072*x^5-3720477830*x^4\
-7708671199*x^3-8651870259*x^2-5234939626*x-1335284456);
Q6(x)=-2^11*(142449224*x^6+2322433504*x^5+8274569043*x^4\
+16490093715*x^3+19150537902*x^2+12192770982*x+3274978808);
Q7(x)=2^13*(981166912*x^6+10733250112*x^5+42252481014*x^4\
+89788613797*x^3+109652862169*x^2+72517376554*x+20047278592);
Q8(x)=-2^18*(372434896*x^6+3658954464*x^5+15311727449*x^4\
+35235218784*x^3+46395190611*x^2+32711646672*x+9519098340);
Q9(x)=2^21*(x+1)*(346136512*x^5+3108047392*x^4+11747918732*x^3\
+23427008330*x^2+24319147839*x+10373546862);
Q10(x)=-2^26*3*(x+1)*(x+2)*(15660944*x^4+128476112*x^3\
+406237252*x^2+587489788*x+324962067);
Q11(x)=2^29*3^3*(x+1)*(x+2)*(x+3)*(480320*x^3+3278592*x^2\
+7590386*x+5789119);
Q12(x)=-2^34*3^4*7*37*(x+1)*(x+2)*(x+3)*(x+4)*(4*x+7)*(4*x+9);
```

There are two solutions that are regular at $z = 0$, with the particular solution $\widetilde{F}_5(z) = 1 + 8z + O(z^2)$ giving the transformation of the Green function. By exact computation of all 9 fold sums in (73) with $k \le 106$, I have verified that the differential equation (74) reproduces the expansion of the Green function up to z^{106}. This took several CPU weeks, spread over a cluster of AMD64 machines running Pari–GP. It would be interesting to know whether the differential equation can be factorized. I remark that it has singularities for $1/z = 0,\ 4,\ \frac{16}{3},\ 8,\ \pm 16,\ 48$ and at the 6 roots of

$$916586496z^6 - 571981824z^5 + 67242496z^4 - 8372096z^3 + 315096z^2 - 6840z + 271 \quad (75)$$

of which only two are real.

4.5 Conjectures for lattice Green functions

I have computed expansions of the Yukawa coupling (65) for the differential equations (60) of Green functions of the diamond lattices in D dimensions, with $10 > D > 3$. For each, I then extracted the numbers n_k of the Lambert series (66), which I denote by $n_k(D)$, to indicate the spatial dimension of the problem. As noted in [4], the sequence $3n_k(4)/k^3$, at $D = 4$, yields integers. However, for $D > 4$, I found no integer $N_0(D)$ such that $N_0(D)n_k(D)/k^3$ invariably yields integers for $k > 0$. Rather, I make the following conjectures.

Conjecture 1: For $D > 3$ and $k > 0$, $n_k(D)/k^2$ is a positive integer.

Conjecture 2: For $k > 0$, $n_k(D)$ is a polynomial in D with degree k. In particular, $n_1(D) = D - 3$, $n_2(D) = 2(D-3)(3D-4)$,

$$
\begin{aligned}
n_3(D) &= 42(D-1)(D-2)(D-3), && (76)\\
n_4(D) &= \frac{8}{3}(D-1)(D-3)(127D^2 - 551D + 588), && (77)\\
n_5(D) &= \frac{5}{6}(D-1)(D-3)(3684D^3 - 26104D^2 + 62237D - 49560), && (78)\\
n_6(D) &= \frac{2}{35}n_3(D)(12786D^3 - 105432D^2 + 302817D - 303400). && (79)
\end{aligned}
$$

Remark 2: Conjecture 1 has been validated for $D < 10$ and $k < 100$. The formulas for $n_k(D)$ with $k < 7$, in Conjecture 2, are consistent with the data for $D < 10$ and with Conjecture 1.

5 Comments and conclusion

I congratulate the authors of [10] for arriving at conjecture (2) on the basis of rather slender evidence, some of which contained significant numerical inaccuracies. Thanks to a simple relation (3) to Bessel moments, proven in Theorem 1 of Section 2, and the excellence of Pari-GP, I have been able to validate the conjecture at 50 digit precision in 100 highly non-trivial cases. Yet this brings us no closer to a proof.

I find it notable that the recursions (14) lead to differential equations (60) whose mirror maps have rather simple Yukawa couplings, as exemplified by Conjectures 1 and 2, based on analyses of diamond lattice Green functions in $D < 10$ spatial dimensions. It is satisying that the FCC lattice Green function for $D = 4$ leads to a differential equation (68) of comparable form, albeit with degree 6, making the original result in [13], with degree 7 and no obvious symmetries, less mysterious. This result is now underwritten to $O(z^{100})$, again thanks to Pari–GP. The far more difficult task of finding a differential equation for the FCC lattice Green function in $D = 5$ dimensions has been completed, with a result in (74) that is sadly lacking MUM. This is underwritten by Pari–GP to $O(z^{106})$, after considerable effort. I urge others to try to factorize it, in case it too may lead to a Calabi–Yau equation, as do the BCC, SC and diamond lattices with $D = 5$.

In conclusion, I echo a remark in [10] that random walks lead to a fascinating blend of probabilistic, analytic, algebraic and combinatorial problems and add to these the connection to self returning walks on lattices and the mirror maps of the differential equations for the associated Green functions.

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